8 SUCCESSIVE DIFFERENTIATION

8.1. Definitions and Notations.

We have seen that the derivative of a function of x, say $f(x)$, is in general a function of x. This new function (*i.e.*, the derivative) may have a derivative, which is called the *second derivative* (*or second differential coefficient*) of $f(x)$, the original derivative being called the *first derivative* (*or first differential coefficient*). Similarly, the derivative of the second derivative is called the *third derivative*; and so on for the *n-th derivative*.

Thus, if $y = x^3$, $\dfrac{dy}{dx} = 3x^2$

Again, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(3x^2\right) = 6x$

Since $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is denoted by $\dfrac{d^2y}{dx^2}$. Therefore, $\dfrac{d^2y}{dx^2}$ (i.e., the second derivative of y with respect to x) in this case is $6x$.

Again, $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d}{dx}(6x) = 6$

Since $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$ is denoted by $\dfrac{d^3y}{dx^3}$.

$\therefore$ $\dfrac{d^3y}{dx^3}$ (i.e., the third derivative of y with respect to x) is 6 here.

Similarly, the n-th derivative of y with respect to x is generally denoted by $\dfrac{d^ny}{dx^n}$.

If $y = f(x)$, the successive derivatives are also denoted by

	y_1,	y_2,	$y_3, \ldots\ldots\ldots$	y_n
or,	y',	y''	$y''', \ldots\ldots\ldots$	$y^{(n)}$
or,	$\dot{y}$,	$\ddot{y}$,	$\dddot{y} \ldots\ldots\ldots$	
or,	$f'(x)$,	$f''(x)$,	$f'''(x) \ldots\ldots\ldots$	$f^{(n)}(x)$
or,	$D\,f(x)$	$D^2 f(x)$,	$D^3 f(x) \ldots\ldots$	$D^n f(x)$

D standing for the symbol $\dfrac{d}{dx}$.

8.2. The *n*-th derivatives* of some special functions.

(i) $\mathbf{y} = \mathbf{x}^n$, where n is a positive integer.

$y_1 = nx^{n-1}$;

$y_2 = n(n-1)x^{n-2}$;

$y_3 = n(n-1)(n-2)x^{n-3}$ and proceeding in a similar manner,

$y_r = n(n-1)(n-2)\ldots\{n-(r-1)\}x^{n-r}$ $(r<n)$

$y_n = n(n-1)(n-2)\ldots 3.2.1 = n!$,

i.e., $\mathbf{D^n(x^n) = n!}$

Cor. Since $y_n = n!$, which is a constant, y_{n+1}, $y_{n+2}, \ldots$ etc. all zeroes in this case.

(ii) $\mathbf{y} = \mathbf{(ax+b)^m}$, where m is any number.

$y_1 = ma(ax+b)^{m-1}$;

$y_2 = m(m-1)a^2(ax+b)^{m-2}$;

$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$; and proceeding similarly,

$y_n = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)a^n(ax+b)^{m-n}$,

$\mathbf{D^n(ax+b)^m = m(m-1)(m-2)(m-n+1)a^n(ax+b)^{m-n}}$.

If m be a positive integer greater than n,

Since $m(m-1)(m-2)\ldots\ldots(m-n+1) = \dfrac{m!}{(m-n)!}$,

$\therefore$ $\mathbf{D^n(ax+b)^m = \dfrac{m!}{(m-n)!}\,a^n(ax+b)^{m-n}}$,

m being a positive integer greater than n.

Note. If m be a positive integer less than n, $D_n(ax+b)^m = 0$

When $m = n$, $D_n(ax+b)^n = a^n.n!$.

(iii) $\mathbf{y = e^{ax}}$

$\therefore$ $y_1 = ae^{ax}$; $y_2 = a^2e^{ax}$; $y_3 = a^3e^{ax}$; .... $y_n = a^ne^{ax}$.

$\therefore$ $\mathbf{D^n\left(e^{ax}\right) = a^ne^{ax}}$.

Cor. (i) $D^n(e^x) = e^x$

Cor. (ii) $y = a^x = e^{x\log_e a}$, $\therefore$ $D^n(a^x) = (\log_e a)^n a^x$.

* Strictly speaking, in these cases, the n-th derivatives are to be established generally by the method of Induction.

(iv) $\mathbf{y = \dfrac{1}{x+a}}$

$\therefore\ y_1 = -1.(x+a)^{-2}$

$y_2 = (-1)(-2)(x+a)^{-3} = (-1)^2.2!.(x+a)^{-3}$

Similarly, $y_3 = (-1)^3.3!.(x+a)^{-4}$ etc.

$$\therefore\ \mathbf{D}^n\left(\frac{1}{x+a}\right) = \frac{(-1)^n\, n!}{(x+a)^{n+1}}.$$

Cor. Proceeding as above, $D^n\left\{\dfrac{1}{(ax+b)^m}\right\} = \dfrac{(-1)^n a^n (m+n-1)!}{(m-1)!\,(ax+b)^{m+n}}$

(v) $\mathbf{y = \log(x+a)}$

$\therefore\ y_1 = \dfrac{1}{x+a}$. Hence, as in (iv) above

$$\mathbf{D}^n\{\log(x+a)\} = \frac{(-1)^{n-1}(n-1)!}{(x+a)^n}.$$

Cor. $D^n\{\log(ax+b)\} = \dfrac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$.

(vi) $\mathbf{y = \sin(ax+b)}$

$y_1 = a\cos(ax+b) = a\sin\left(\tfrac{1}{2}\pi + ax + b\right)$

$y_2 = a^2\cos\left(\tfrac{1}{2}\pi + ax + b\right) = a^2\sin\left(2.\tfrac{1}{2}\pi + ax + b\right)$

$y_3 = a^3\cos\left(2.\tfrac{1}{2}\pi + ax + b\right) = a^3\sin\left(3.\tfrac{1}{2}\pi + ax + b\right)$ etc

$$\therefore\quad \mathbf{D}^n\{\sin(ax+b)\} = a^n\sin\left(\frac{n}{2}\pi + ax + b\right)$$

Similarly, $\mathbf{D}^n\{\cos(ax+b)\} = a^n\cos\left(\dfrac{n}{2}\pi + ax + b\right)$

As particular cases when $b = 0$,

$$\mathbf{D}^n\{\sin ax\} = a^n\sin\left(\frac{n}{2}\pi + ax\right);$$

$$\mathbf{D}^n\{\cos ax\} = a^n\cos\left(\frac{n}{2}\pi + ax\right).$$

8.3. The n-th derivatives of rational algebraic functions.

The nth derivative of a fraction whose numerator and denominator are both rational integral algebraic functions may be conveniently obtained by resolving the fraction into *partial fractions*. This is shown in Ex. 4, Art. 8.4. The rules for decomposing a fraction into partial fractions are given in the Appendix.

Even when the denominator of a given algebraic fraction cannot be broken up into *real* linear factors, the above method of decomposition can be used by resolving the denominator into *imaginary* linear factors. In this case, *DeMoivre's theorem* is conveniently applied to put the final result in the real form. This is illustrated in Ex. 5, Art. 8.4.

8.4. Illustrative Examples.

Ex. 1. *If* $y = \sin^3 x$, *find* y_n.

$$\sin 3x = 3\sin x - 4\sin^3 x.$$

$$\therefore\quad y = \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$\therefore\quad y_n = \frac{1}{4}\left\{3\sin\left(\frac{1}{2}n\pi + x\right) - 3^n\sin\left(\frac{1}{2}n\pi + 3x\right)\right\}$$

Ex. 2. *If* $y = \sin 3x.\cos 2x$, *find* y_n.

$$y = \frac{1}{2}\cdot 2\sin 3x\cos 2x = \frac{1}{2}(\sin 5x + \sin x)$$

$$\therefore\ y_n = \frac{1}{2}\left\{5^n\sin\left(\frac{1}{2}n\pi + 5x\right) + \sin\left(\frac{1}{2}n\pi + x\right)\right\}.$$

Ex. 3. *If* $y = e^{ax}\sin bx$, *find* y_n.

$$y_1 = e^{ax}.a\sin bx + e^{ax}.\cos bx.b$$

$$= e^{ax}.(a\sin bx + b\cos bx).$$

Let $a = r\cos\phi$, $b = r\sin\phi$, so that

$$r = \left(a^2 + b^2\right)^{\frac{1}{2}},\ \phi = \tan^{-1}\frac{b}{a}$$

$$\therefore\ y_1 = re^{ax}\sin(bx + \phi).$$

Similarly, $y_2 = re^{ax}\{a\sin(bx+\phi) + b\cos(bx+\phi)\}$

$= r^2e^{ax}\sin(bx + 2\phi)$, as before.

In a similar way $y_3 = r^3 e^{ax} \sin(bx + 3\phi)$, etc. and generally

$$y_n = r^n e^{ax} \sin(bx + n\phi),$$

i.e., $D^n(e^{ax} \sin bx) = \left(a^2 + b^2\right)^{\frac{1}{2}n} e^{ax} \sin\left(bx + n\tan^{-1}\dfrac{b}{a}\right)$,

Note. Similarly,

$$D^n(e^{ax} \cos bx) = \left(a^2 + b^2\right)^{\frac{1}{2}n} e^{ax} \cos\left(bx + n\tan^{-1}\frac{b}{a}\right),$$

Again, if $y = e^{ax} \sin(bx + c)$,

$$y_n = \left(a^2 + b^2\right)^{\frac{n}{2}} e^{ax} \sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$$

and if $y = e^{ax} \cos(bx + c)$,

$$y_n = \left(a^2 + b^2\right)^{\frac{n}{2}} e^{ax} \cos\left(bx + c + n\tan^{-1}\frac{b}{a}\right).$$

Ex. 4. *If* $y_n = \dfrac{x^2 + x - 1}{x^3 + x^2 - 6x}$, *find* y_n.

$$x^3 + x^2 - 6x = x(x^2 + x - 6) = x(x+3)(x-2)$$

Let $\dfrac{x^2 + x - 1}{x^3 + x^2 - 6x} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{x-2}$.

Multiplying both sides by $x(x+3)(x-2)$, we get

$$x^2 + x - 1 = A(x+3)(x-2) + Bx(x-2) + Cx(x+3).$$

Putting $x = 0, -3, 2$ successively on both sides, we get

$$A = \frac{1}{6}, \quad B = \frac{1}{3}, \quad C = \frac{1}{2}.$$

$$\therefore \quad y = \frac{1}{6}\cdot\frac{1}{x} + \frac{1}{3}\cdot\frac{1}{x+3} + \frac{1}{2}\cdot\frac{1}{x-2}.$$

$$\therefore \quad y_n = (-1)^n n! \left\{\frac{1}{6}\cdot\frac{1}{x^{n+1}} + \frac{1}{3}\cdot\frac{1}{(x+3)^{n+1}} + \frac{1}{2}\cdot\frac{1}{(x-2)^{n+1}}\right\}.$$

Ex. 5. *If* $y = \dfrac{1}{x^2 + a^2}$ *find* y_n.

$$y = \frac{1}{(x + ia)(x - ia)} = \frac{1}{2ia}\left(\frac{1}{x - ia} - \frac{1}{x + ia}\right)$$

$$\therefore \quad y_n = \frac{(-1)^n n!}{2ia}\left\{\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right\}$$

$$= \frac{(-1)^n n!}{2ia}\left\{(x-ia)^{-(n+1)} - (x+ia)^{-(n+1)}\right\}.$$

Put $x = r\cos\theta, \quad a = r\sin\theta$

So that $r = (x^2 + a^2)^{\frac{1}{2}}, \quad \theta = \tan^{-1} a/x.$

Now, $(x-ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta - i\sin\theta)^{-(n+1)}$

$$= r^{-(n+1)}\{\cos(n+1)\theta + i\sin(n+1)\theta\}$$

$$(x+ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta + i\sin\theta)^{-(n+1)}$$

$$= r^{-(n+1)}\{\cos(n+1)\theta - i\sin(n+1)\theta\}$$

$$\therefore \quad y_n = \frac{(-1)^n n!}{2ia}\cdot r^{-(n+1)}.2i\sin(n+1)\theta.$$

Since $r = \dfrac{a}{\sin\theta}$, $r^{-(n+1)} = \dfrac{a^{-(n+1)}}{\sin^{-(n+1)}\theta} = \dfrac{\sin^{n+1}\theta}{a^{n+1}}$

$$\therefore D^n\left(\frac{1}{x^2 + a^2}\right) = \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\ \sin(n+1)\theta,$$

where $\theta = \tan^{-1}\dfrac{a}{x} = \cot^{-1}\dfrac{x}{a}$.

Note. If $y = \dfrac{1}{(x+b)^2 + a^2}$, $y_n = \dfrac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$,

where $\theta = \tan^{-1}\{a/(b+x)\} = \cot^{-1}\{(b+x)/a\}$.

Cor. If $y = \tan^{-1}x$, $y_1 = \dfrac{1}{1+x^2}$ hence

$$\mathbf{D^n\left(\tan^{-1}x\right) = (-1)^{n-1}(n-1)!\sin^n\theta\ \sin n\theta}.$$

where $\theta = \tan^{-1}\dfrac{1}{x} = \cot^{-1}x$.

EXAMPLES - VIII (A)

1. Find y_n in the following cases :

(i) $y = (a - bx)^m$. (ii) $y = 1/(ax+b)^m$.

(iii) $y = 1/(a-x)$. (iv) $y = \log(ax+b)^p$.

(v) $y = \log\{(a-x)/(a+x)\}$. (vi) $y = \sqrt{x}$.

(vii) $y = 1/\sqrt{x}$. (viii) $y = (2-3x)^n$.

(ix) $y = \log(ax + x^2)$ (x) $y = 10^{3-2x}$.

(xi) $y = x/(a+bx)$. (xii) $y = (a-x)/(a+x)$.

(xiii) $y = x^n/(x-1)$. (xiv) $y = \sin^2 x$.

(xv) $y = \cos 2x \cos x$. (xvi) $y = \cos^3 x$.

(xvii) $y = \sin^2 x \cos^2 x$. (xviii) $y = \sin x \sin 2x \sin 3x$.

(xix) $y = e^x \cos x$. (xx) $y = e^x \sin x \sin 2x$.

(xxi) $y = e^{3x} \sin 4x$. (xxii) $y = e^x \sin^2 x$.

2. Find y_3, if

(i) $y = x^2 \log x$. (ii) $y = e^{\sin x}$.

(iii) $y = e^{1/x}$. (iv) $y = \sin^{-1} x$.

3. Find the n-th derivatives of the following functions :

(i) $\dfrac{1}{x^2 - a^2}$. (ii) $\dfrac{1}{x^2+16}$. [C.P.1993] (iii) $\tan^{-1}\dfrac{x}{a}$.

(iv) $\dfrac{x}{x^2+a^2}$. (v) $\dfrac{1}{x^4 - a^4}$. (vi) $\dfrac{1}{x^2+x+1}$.

(vii) $\dfrac{1}{(x^2+a^2)(x^2+b^2)}$. (viii) $\dfrac{1}{4x^2+4x+5}$.

(ix) $\dfrac{x^4}{(x-1)(x-2)}$. (x) $\dfrac{x^2+1}{(x-1)(x-2)(x-3)}$. [C.P. 1993]

(xi) $\dfrac{x^2}{(x+1)^2(x+2)}$. (xii) $\dfrac{x^2}{(x-a)(x-b)}$.

(xiii) $\tan^{-1}\dfrac{1+x}{1-x}$. (xiv) $\sin^{-1}\dfrac{2x}{1+x^2}$.

(xv) $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$. (xvi) $\cot^{-1}\dfrac{x}{a}$.

4. If $y = x^{2n}$, where n is a positive integer, show that
$$y_n = 2^n\{1.3.5\ldots(2n-1)\}x^n.$$

5. If $u = \sin ax + \cos ax$, show that
$$u_n = a^n\{1 + (-1)^n \sin 2ax\}^{1/2}.$$ [B. P. 1993]

6. If $ax^2 + 2hxy + by^2 = 1$, show that
$$\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx+by)^3}.$$

7. Find y_2, if

(i) $\sin x + \cos y = 1$.

(ii) $y = \tan(x+y)$.

(iii) $x^3 + y^3 - 3axy = 0$.

8. (a) Find $\dfrac{d^2y}{dx^2}$ in the following cases :

(i) If $x = a\cos\theta$, $y = b\sin\theta$.

(ii) If $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. [C. P. 1988]

(b) If $x = \cos t$ and $y = \log t$, then prove that at $t = \dfrac{\pi}{2}$
$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$$ [J.E.E. 1985]

9. If $x = f(t)$, $y = \phi(t)$, then
$$\frac{d^2y}{dx^2} = \frac{x_1y_2 - y_1x_2}{x_1^3}$$ [V. P. 1998]

where suffixes denote differentiations with respect to t.

10. If $x\sin\theta + y\cos\theta = a$ and $x\cos\theta - y\sin\theta = b$ then prove that
$$\frac{d^px}{d\theta^p}\cdot\frac{d^qy}{d\theta^q} - \frac{d^qx}{d\theta^q}\cdot\frac{d^py}{d\theta^p}$$ is a constant.

11. Show that $\dfrac{d^n}{dx^n}\left(\dfrac{1}{x^2+1}\right) = \dfrac{(-1)^n n!}{(x^2+1)^{n+1}} \times$
$$\left\{(n+1)x^n - \binom{n+1}{3}x^{n-2} + \binom{n+1}{5}x^{n-4} - \ldots\right\}.$$

12. If $y = \sin mx$, show that

$$\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix} = 0,$$

where suffixs of y denote the order of differentiations of y with respect to x.

ANSWERS

1. (i) $(-1)^n m(m-1)(m-2)\ldots(m-n+1)\, b^n (a-bx)^{m-n}$

(ii) $\dfrac{(-1)^n m(m-1)(m-2)\ldots(m-n+1)\, a^n}{(a+bx)^{m+n}}$

(iii) $\dfrac{n!}{(a-x)^{n+1}}$. (iv) $\dfrac{(-1)^{n-1} p.a^n (n-1)!}{(ax+b)^n}$

(v) $(n-1)!\left\{\dfrac{-1}{(a-x)^n} + \dfrac{(-1)^n}{(a+x)^n}\right\}$. (vi) $(-1)^{n-1} \cdot \dfrac{1.3.5\ldots(2n-3)}{2^n x^{n-\frac{1}{2}}}$.

(vii) $(-1)^n \cdot \dfrac{1.3.5\ldots(2n-1)}{2^n x^{n+\frac{1}{2}}}$. (viii) $(-1)^n . 3^n . n!$

(ix) $(-1)^{n-1}(n-1)!\left\{\dfrac{1}{x^n} + \dfrac{1}{(a+x)^n}\right\}$. (x) $10^{3-2x}.(-2)^n.(\log_e 10)^n$

(xi) $\dfrac{(-1)^{n+1} ab^{n-1} n!}{(a+bx)^{n+1}}$. (xii) $\dfrac{2a(-1)^n n!}{(a+x)^{n+1}}$.

(xiii) $\dfrac{(-1)^n n!}{(x-1)^{n+1}}$. (xiv) $-2^{n-1}\cos\left(\dfrac{1}{2}n\pi + 2x\right)$

(xv) $\dfrac{1}{2}\left\{3^n \cos\left(\dfrac{1}{2}n\pi + 3x\right) + \cos\left(\dfrac{1}{2}n\pi + x\right)\right\}$.

(xvi) $\dfrac{1}{4}\left\{3\cos\left(\dfrac{1}{2}n\pi + x\right) + 3^n \cos\left(\dfrac{1}{2}n\pi + 3x\right)\right\}$.

(xvii) $-2^{2n-2}\cos\left(\dfrac{1}{2}n\pi + 4x\right)$

(xviii) $\dfrac{1}{4}\left\{4^n \sin\left(\dfrac{1}{2}n\pi + 4x\right) + 2^n \sin\left(\dfrac{1}{2}n\pi + 2x\right) - 6^n . \sin\left(\dfrac{1}{2}n\pi + 6x\right)\right\}$

(xix) $2^{\frac{1}{2}n} e^x \cos\left(x + \dfrac{1}{4}n\pi\right)$

(xx) $\dfrac{1}{2}e^x\left\{2^{\frac{1}{2}n}\cos\left(x + \dfrac{1}{4}n\pi\right) - 10^{\frac{1}{2}n}\cos\left(3x + n\tan^{-1}3\right)\right\}$.

(xxi) $5^n e^{3x}\sin\left(4x + n\tan^{-1}\dfrac{4}{3}\right)$

(xxii) $\dfrac{1}{2}e^x\left\{1 - 5^{\frac{1}{2}n}\cos\left(2x + n\tan^{-1}2\right)\right\}$.

2. (i) $2/x$ (ii) $-e^{\sin x}\cos x . \sin x . (\sin x + 3)$

(iii) $-\dfrac{\left(1+6x+6x^2\right)e^{\frac{1}{x}}}{x^6}$. (iv) $\dfrac{\left(1+2x^2\right)}{\left(1-x^2\right)^{5/2}}$.

3. (i) $\dfrac{(-1)^n n!}{2a}\left\{\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(x+a)^{n+1}}\right\}$.

(ii) $\dfrac{(-1)^n n! \sin^{n+1}\theta \sin(n+1)\theta}{4^{n+2}}$, where $\theta = \tan^{-1}\dfrac{4}{x}$.

(iii) $\dfrac{(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta}{a^n}$, where $\theta = \tan^{-1}\dfrac{a}{x}$.

(iv) $\dfrac{(-1)^n n!}{a^{n+1}}\sin^{n+1}\theta \cos(n+1)\theta$, where $\theta = \tan^{-1}\dfrac{a}{x}$.

(v) $\dfrac{(-1)^n n!}{4a^3}\left\{\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(x+a)^{n+1}} - \dfrac{2}{a^{n+1}}\sin^{n+1}\theta \sin(n+1)\theta\right\}$

where $x = a\cot\theta$.

(vi) $\dfrac{(-1)^n . 2^{n+2} n!}{\left(\sqrt{3}\right)^{n+2}}\sin^{n+1}\theta \sin(n+1)\theta$, where $\theta = \tan^{-1}\left\{\sqrt{3}/(2x+1)\right\}$.

(vii) $\dfrac{(-1)^n n!}{a^2 - b^2}\left\{\dfrac{\sin^{n+1}\theta \sin(n+1)\theta}{b^{n+2}} - \dfrac{\sin^{n+1}\phi \sin(n+1)\phi}{a^{n+2}}\right\}$

(viii) $(-1)^n n! \dfrac{1}{4}\sin^{n+1}\theta \sin(n+1)\theta$, where $\cot\theta = x + \dfrac{1}{2}$

(ix) $(-1)^n n!\left\{\dfrac{16}{(x-2)^{n+1}} - \dfrac{1}{(x-1)^{n+1}}\right\}$. when $n > 2$.

(x) $(-1)^n n!\left\{\dfrac{16}{(x-1)^{n+1}}-\dfrac{5}{(x-2)^{n+1}}+\dfrac{5}{(x-3)^{n+1}}\right\}$.

(xi) $(-1)^n n!\left\{\dfrac{n+1}{(x+1)^{n+2}}-\dfrac{3}{(x+1)^{n+1}}+\dfrac{4}{(x+2)^{n+1}}\right\}$.

(xii) $\dfrac{(-1)^n n!}{(a-b)}\left\{\dfrac{a^2}{(x-a)^{n+1}}-\dfrac{b^2}{(x+a)^{n+1}}\right\}$.

(xiii) $(-1)^{n-1}(n-1)!\sin^n\theta\ \sin n\theta$, where $\cot\theta = x$.

(xiv) $2(-1)^{n-1}(n-1)!\sin^n\theta\ \sin n\theta$, where $\cot\theta = x$.

(xv) $\dfrac{1}{2}(-1)^{n-1}(n-1)!\sin^n\theta\ \sin n\theta$, where $\cot\theta = x$.

(xvi) $\dfrac{(-1)^n (n-1)!\ \sin^n\theta\ \sin n\theta}{a^n}$, where $\theta = \cot^{-1}\dfrac{x}{a}$.

7. (i) $-\dfrac{\sin^2 x+\cos y}{\sin^3 y}$ (ii) $-\dfrac{2(1+y^2)}{y^5}$. (iii) $-\dfrac{2a^3xy}{(y^2-ax)^3}$.

8. (a) (i) $-\dfrac{b}{a^2}\operatorname{cosec}^3\theta$ (ii) $\dfrac{1}{4a}\sec^4\dfrac{\theta}{2}$

8.5. Leibnitz's Theorem*. (*n-th derivative of the product of two functions*)

If u and v are two functions of x, each possessing derivatives upto nth order, then the nth derivative of their product, i.e.,

$$(uv)_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \ldots + {}^nc_r u_{n-r} v_r + \ldots + uv_n,$$

where the suffixes of u and v denote the order of differentiations of u and v with respect to x.

Let $y = uv$.

By actual differentiation, we have

$$y_1 = u_1 v + uv_1,$$

$$y_2 = u_2 v + 2u_1 v_1 + uv_2 = u_2 v + {}^2c_1 u_1 v_1 + uv_2,$$

$$y_3 = u_3 v + 3u_2 v_1 + 3u_1 v_2 + uv_3,$$

$$= u_3 v + {}^3c_1 u_2 v_1 + {}^3c_2 u_1 v_2 + uv_3,$$

The theorem is thus seen to be true when $n = 2$ and 3.

* Leibnitz (1646-1716) was a German mathematician, who invented Calculus in Germany, as Newton did in England.

Let us *assume*, therefore, that

$$y_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + \ldots + {}^n c_r u_{n-r} v_r + \ldots + uv_n,$$

where n has any particular value.

$\therefore$ differentiating,

$$y_{n+1} = u_{n+1} v + ({}^n c_1 + 1) u_n v_1 + ({}^n c_2 + {}^n c_1) u_{n-1} v_2 + \ldots$$

$$+ ({}^n c_r + {}^n c_{r-1}) u_{n-r+1} v_r + \ldots + uv_{n+1},$$

Since ${}^n c_r + {}^n c_{r-1} = {}^{n+1} c_r$ and ${}^n c_1 + 1 = {}^{n+1} c_1$

$$\therefore\quad y_{n+1} = u_{n+1} v + {}^{n+1} c_1 u_n v_1 + {}^{n+1} c_2 u_{n-1} v_2 + \ldots$$

$$+ {}^{n+1} c_r u_{n-r+1} v_r + \ldots + uv_{n+1}.$$

Thus, if the theorem holds for n differentiations, it also holds for $n+1$. But it is proved to hold for 2 and 3 differentiations; hence it holds for four, and so on, and thus the therem is true for every positive integral value of n.

8.6. Important results of symbolic operation.

If $F(D)$ be any rational integral algebraic function of D or $\dfrac{d}{dx}$ (the symbolic operator), i.e., if

$$F(D) = A_n D^n + A_{n-1} D^{n-1} + \ldots\ldots + A_1 D + A$$

$$= \sum A_r D^r, \text{ where } A_r \text{ is independent of } D, \text{ then}$$

(i) $F(D)e^{ax} = F(a)e^{ax}$.

(ii) $F(D)\, e^{ax} V = e^{ax} F(D+a)\, V$, *V being a function of x.*

(iii) $F(D^2)\begin{cases}\sin(ax+b)\\ \cos(ax+b)\end{cases} = F(-a^2)\begin{cases}\sin(ax+b)\\ \cos(ax+b)\end{cases}$

Proof:

(i) Since $D^r e^{ax} = a^r e^{ax}$,

$$\therefore\ F(D)e^{ax} = \sum A_r D^r e^{ax}$$

$$= \sum A_r a^r e^{ax}$$

$$= \left(\sum A_r \cancel{D^r}\right) e^{ax}$$

$$= F(a) e^{ax}$$

(ii) Let $y=e^{ax}V$. Since $D^r e^{ax}=a^r e^{ax}$,

$\therefore$ by Leibnitz's Theorem, we have

$$y_n = e^{ax}\left(a^nV + {}^nc_1a^{n-1}DV + {}^nc_2a^{n-2}D^2V + ... + D^nV\right)$$

which by analogy with the Binomial theorem may be written as

$$D^n\left(e^{ax}V\right) = e^{ax}(D+a)^nV,$$

$$F(D)\, e^{ax}V = \left(\sum A_rD^r\right)e^{ax}V$$

$$= \sum A_rD^re^{ax}V$$

$$= e^{ax}\sum A_r(D+a)^rV$$

$$= e^{ax}F(D+a)V.$$

(iii) We have $D\sin(ax+b) = a\cos(ax+b)$, and so on

$$D^2\sin(ax+b) = \left(-a^2\right)\sin(ax+b);$$

$$D^{2r}\sin(ax+b) = \left(-a^2\right)^r\sin(ax+b).$$

Hence, as in (i) and (ii), it follows that

$$F\left(D^2\right)\sin(ax+b) = F\left(-a^2\right)\sin(ax+b)$$

Similarly, $F\left(D^2\right)\cos(ax+b) = F\left(-a^2\right)\cos(ax+b)$

8.7 Illustrative Examples.

Ex. 1. *If* $y=e^{ax}x^3$, *find* y_n.

Let $u=e^{ax}$, $v=x^3$. Now, $u_n=a^ne^{ax}$.

$\therefore$ by Leibnitz's Theorem,

$$y_n = a^ne^{ax}x^3 + n.a^{n-1}e^{ax}.3x^2 + \frac{n(n-1)}{2!}.a^{n-2}e^{ax}.6x + \frac{n(n-1)(n-2)}{3!}.a^{n-3}e^{ax}.6$$

$$= e^{ax}a^{n-3}\left\{a^3x^3 + 3na^2x^2 + 3n(n-1)ax + n(n-1)(n-2)\right\}$$

Ex. 2. *If* $y = a\cos(\log x) + b\sin(\log x)$, *show that* $x^2y_2 + xy_1 + y = 0$.

Differentiating,

$$y_1 = -a\sin(\log x).\frac{1}{x} + b\cos(\log x).\frac{1}{x};$$

$$\therefore\ xy_1 = -a\sin(\log x) + b\cos(\log x).$$

Differentiating again,

$$xy_2 + y_1 = -a\cos(\log x).\frac{1}{x} - b\sin(\log x).\frac{1}{x},$$

$$\therefore\ x^2y_2 + xy_1 = -(a\cos\log x + b\sin\log x) = -y,$$

$$\therefore\ x^2y_2 + xy_1 + y = 0.$$

Note. This is called the differential equation formed from the above equation.

Ex. 3. *Differentiate n times the equation*

$$(1+x^2)y_2 + (2x-1)y_1 = 0.$$

By Leibnitz's Theorem,

$$\frac{d^n}{dx^n}\{y_2(1+x^2)\} = y_{n+2}(1+x^2) + n.y_{n+1}.2x + \frac{n(n-1)}{21}y_n 2$$

$$\frac{d^n}{dx^n}\{y_2(2x-1)\} = y_{n+1}(2x-1) + n.y_n.2$$

adding,

$$(1+x^2)y_{n+2} + \{2(n+1)x-1\}y_{n+1} + n(n+1)y_n = 0.$$

Ex. 4. *Find the value of* y_n *for* $x=0$, *when* $y=e^{a\sin^{-1}x}$.

From the value of y, when $x=0$, $y=1$

Here $y_1 = e^{a\sin^{-1}x}.a\frac{1}{\sqrt{1-x^2}}$ ... (1)

$$= ay\frac{1}{\sqrt{1-x^2}}.$$

$$\therefore\quad y_1^2\left(1-x^2\right) = a^2y^2.$$

Differentiating, $2y_1y_2\left(1-x^2\right) + y_1^2(-2x) = 2a^2yy_1$,

or, $\left(1-x^2\right)y_2 - xy_1 - a^2y = 0.$ ... (2)

Differentiating this n times by Leibnitz's Theorem as in Ex. 3, we easily get $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2+a^2\right)y_n = 0.$

Putting $x=0$, $(y_{n+2})_0 = \left(n^2+a^2\right)(y_n)_0.$ ... (3)

Replacing n by $n-2$, we get, similarly

$$(y_n)_0 = \{(n-2)^2 + a^2\}(y_{n-2})_0$$

$$= \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}(y_{n-4})_0$$

Also from (1) and (2), $(y_1)_0 = a$, $\quad (y_2)_0 = a^2$.

Thus $(y_n)_0 = \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}\ldots$

$$(4^2 + a^2)(2^2 + a^2)a^2 \text{, if } n \text{ is even}$$

and $= \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}\ldots$

$$(3^2 + a^2)(1^2 + a^2)a^2 \text{, if } n \text{ is odd.}$$

Note. The value of y_n for $x = 0$ is shortly denoted by $(y_n)_0$.

8.12 Miscellaneous Worked Out Examples

Ex. 1. (*i*) If $F(x) = f(x)\,\phi(x)$ and $f'(x)\,\phi'(x) = k$, (k is a constant), then show that

$$\frac{F''}{F} = \frac{f''}{f} + \frac{\phi''}{\phi} + \frac{2k}{f\phi}, \qquad (F(x) \neq 0)$$

(*ii*) If $x = \sin t$, $y = \sin kt$ where k is a constant, show that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + k^2 y = 0.$$

Solution : (*i*) $\because$ $F(x) = f(x)\,\phi(x)$,

$$F'(x) = f'(x)\,\phi(x) + f(x)\,\phi'(x) \qquad \ldots \quad (1)$$

Differentiating (1) w.r.t. x,

$$F''(x) = f''(x)\,\phi(x) + f'(x)\,\phi'(x) + f'(x)\,\phi'(x) + f'(x)\,\phi''(x)$$

$$= f''(x)\,\phi(x) + f(x)\,\phi''(x) + 2k$$

$\because$ $F(x) = f(x)\,\phi(x) \neq 0$, dividing the left-hand side by $F(x)$ and the right-hand side by $f(x)\ \phi(x)$, we get

$$\frac{F''(x)}{F(x)} = \frac{f''(x)}{f(x)} + \frac{\phi''(x)}{\phi(x)} + \frac{2k}{f(x)\,\phi(x)}$$

i.e., $\dfrac{F''}{F} = \dfrac{f''}{f} + \dfrac{\phi''}{\phi} + \dfrac{2k}{f\,\phi}$.

(*ii*) $\because$ $x = \sin t$, $\quad \dfrac{dx}{dt} = \cos t$

and $y = \sin kt$, $\quad \dfrac{dy}{dt} = k\cos kt$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{k\cos kt}{\cos t}$

and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{k\cos kt}{\cos t}\right)\cdot\dfrac{dt}{dx}$

$$= \frac{\cos t\left(-k^2\sin kt\right) - k\cos kt(-\sin t)}{\cos^2 t}\cdot\frac{1}{\cos t}$$

$$= \frac{\{-k^2\cos t\sin kt + k\sin t\cdot\cos kt\}}{\left(1-\sin^2 t\right)\cos t}$$

$$= \frac{1}{1-x^2}\cdot\left\{-k^2\sin kt + \sin t\left(\frac{k\cos kt}{\cos t}\right)\right\}$$

$$= \frac{1}{1-x^2}\cdot\left\{-k^2 y + x\frac{dy}{dx}\right\}$$

or, $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + k^2 y = 0$

Ex. 2. If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that $(x^2-1)y_2 + xy_1 - m^2 y = 0$,

where, $y_1 = \dfrac{dy}{dx}$, $\quad y_2 = \dfrac{d^2y}{dx^2}$.

Solution : $\because$ $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, $\quad a^2 - 2ax + 1 = 0$, where $a = y^{\frac{1}{m}}$

$\therefore$ $a = \dfrac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$

or, $y^{\frac{1}{m}} = x \pm \sqrt{x^2-1}$

Taking logarithm of both the sides,

$$\frac{1}{m}\cdot\log y = \log\left(x \pm \sqrt{x^2-1}\right)$$

Differentiaing both the sides w. r. t. x,

$$\frac{1}{m}\cdot\frac{1}{y}\cdot y_1=\frac{1}{\left(x\pm\sqrt{x^2-1}\right)}\times\left\{1\pm\frac{x}{\sqrt{x^2-1}}\right\}\quad\text{where } y_1=\frac{dy}{dx}$$

or, $\dfrac{y_1}{my}=\pm\dfrac{1}{\sqrt{x^2-1}}$

or, $\left(x^2-1\right)y_1^2=m^2y^2$

Differentiating again w.r.t. x,

$$\left(x^2-1\right)2y_1y_2+2x\cdot y_1^2=m^2\cdot 2y\cdot y_1,\ \text{where } y_2=\frac{d^2y}{dx^2}$$

or, $\left(x^2-1\right)y_2+xy_1-m^2y=0, \qquad 2y_1\neq 0.$

Ex. 3. (*i*) $f\ \ x=a(\theta+\sin\theta),\ \ y=a(1-\cos\theta)$ verify that

$$\frac{d^2y}{dx^2}=\frac{1}{4a}\cdot\sec^4\frac{\theta}{2}.$$ [*C. P. 1988*]

(*ii*) If $x=2\cos\theta-\cos 2\theta,\ \ y=2\sin\theta+\sin 2\theta$ find $\dfrac{d^2y}{dx^2}$ at $\theta=\dfrac{\pi}{2}$.

[*C. P. 1990*]

Solution : (*i*) $\because\ x=a(\theta+\sin\theta),\ \dfrac{dx}{d\theta}=a(1+\cos\theta)$

and $y=a(1-\cos\theta),\quad \dfrac{dy}{d\theta}=a\sin\theta$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}=\frac{a\sin\theta}{a(1+\cos\theta)}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}=\tan\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2}=\frac{dy}{dx}\left(\tan\frac{\theta}{2}\right)=\frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx}$$

$$=\frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{1}{a(1+\cos\theta)}$$

$$=\frac{1}{4a}\cdot\sec^4\frac{\theta}{2}.$$

(*ii*) $\because\ x=2\cos\theta-\cos 2\theta,\quad \dfrac{dx}{d\theta}=2\sin 2\theta-2\sin\theta$

and $\because\ y=2\sin\theta+\sin 2\theta,\quad \dfrac{dy}{d\theta}=2\cos\theta+2\cos 2\theta$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=\frac{2(\cos 2\theta+\cos\theta)}{2(\sin 2\theta-\sin\theta)}=\frac{2\cos\frac{3\theta}{2}\cdot\cos\frac{\theta}{2}}{2\cos\frac{3\theta}{2}\cdot\sin\frac{\theta}{2}}$$

or, $\dfrac{dy}{dx}=\cot\dfrac{\theta}{2}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\cot\frac{\theta}{2}\right)=\frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx}$$

$$=\frac{-\operatorname{cosec}^2\frac{\theta}{2}}{2\cdot 2(\sin 2\theta-\sin\theta)}$$

at $\theta=\dfrac{\pi}{2},\ \dfrac{d^2y}{dx^2}=\dfrac{-\operatorname{cosec}^2\frac{\pi}{4}}{4\left(\sin\pi-\sin\frac{\pi}{2}\right)}=\dfrac{1}{2}.$

Ex. 4. (*i*) If $\sin x=\dfrac{2t}{1+t^2},\ \cot y=\dfrac{1-t^2}{2t}$, find the value of $\dfrac{d^2x}{dy^2}$.

[*C. P. 1987*]

(*ii*) If $y=\dfrac{x}{x+1}$, show that $y_5(0)=5!$ [*C. P. 1991*]

(*iii*) If $y=2\cos x\,(\sin x-\cos x)$, show that $(y_{10})_0=2^{10}$.

[*C. P. 1992, 2000, 2002*]

Solution : (*i*) $\because\ \sin x=\dfrac{2t}{1+t^2}$,

$$x=\sin^{-1}\frac{2t}{1+t^2}=2\tan^{-1}(t) \qquad \ldots (1)$$

$\because\ \cot y=\dfrac{1-t^2}{2t},\quad \tan y=\dfrac{2t}{1-t^2}$

$$\text{or, } y=\tan^{-1}\frac{2t}{1-t^2}=2\tan^{-1}(t) \qquad \ldots (2)$$

From (1) and (2) $x = y$

$$\therefore \frac{dx}{dy} = 1 \text{ and } \frac{d^2x}{dy^2} = 0.$$

(*ii*) $y = \dfrac{x}{x+1} = \dfrac{x+1-1}{x+1} = 1-(x+1)^{-1}$

$$y_1 = (-1)(-1)(x+1)^{-2}$$

$$y_2 = (-1)(-1)(-2)(x+1)^{-3} = (-1)^3 2!(x+1)^{-3}$$

$$y_3 = (-1)^4\, 3!(x+1)^{-4}$$

Proceeding in this way, we have

$$y_5 = (-1)^6\, 5!(x+1)^{-6}$$

$$\therefore \quad (y_5)_0 = (-1)^6\, 5!(1)^{-6} = 5!$$

(*iii*) $y = 2\cos x(\sin x - \cos x)$

$$= 2\sin x\cos x - 2\cos^2 x$$

$$= \sin 2x - (1+\cos 2x)$$

$$= \sin 2x - \cos 2x - 1$$

$$y_{10} = 2^{10}\sin\left\{10\cdot\frac{\pi}{2}+2x\right\} - 2^{10}\cos\left\{10\cdot\frac{\pi}{2}+2x\right\}$$

[vide art. 5·2 (*vi*)]

If $y = \sin ax,\ y_n = a^n\cdot\sin\left(n\dfrac{\pi}{2}+x\right)$, and

if $y = \cos ax,\ y_n = a^n\cdot\cos\left(n\dfrac{\pi}{2}+x\right)$

$$\therefore (y_{10})_0 = 2^{10}\cdot\sin 5\pi - 2^{10}\cos 5\pi = 2^{10}\cdot 0 - 2^{10}\times(-1) = 2^{10}.$$

Ex. 5. If $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, $|x|<1$, show that

(*i*) $(1-x^2)y_2 - 3x\,y_1 - y = 0$

(*ii*) $(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0$ [*C. P. 1993*]

Solution : $\because \quad y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

$$(1-x^2)y^2 = (\sin^{-1}x)^2$$

Differentiating w.r.t. x.

$$(1-x^2)2yy_1 - 2xy^2 = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} = 2y$$

or, $(1-x^2)y_1 - xy = 1 \qquad (\because\ 2y \neq 0)$

Differentiating again w. r. t. x.,

$$(1-x^2)y_2 - 2xy_1 - y - xy_1 = 0$$

or, $(1-x^2)y_2 - 3xy_1 - y = 0$ . . . (1)

Differentiating (1) n times by Leibnitz's theorem,

$$y_{n+2}(1-x^2) + n\cdot y_{n+1}(-2x) + \frac{n(n-1)}{1\cdot 2}\cdot y_n \quad (-2)$$

$$-3\{y_{n+1}\cdot x + n\cdot y_n\cdot 1\} - y_n = 0$$

or, $(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0.$

Ex. 6. If $y = \cos(10\cos^{-1}x)$, show that $(1-x^2)y_{12} = 21xy_{11}$

[*C. P. 1983*]

Solution : $y = \cos(10\cos^{-1}x)$ . . . (1)

$$y_1 = -\sin(10\cos^{-1}x)\frac{(-10)}{\sqrt{1-x^2}} = \frac{10\sin(10\cos^{-1}x)}{\sqrt{1-x^2}}$$

$$(1-x^2)y_1^2 = 100\sin^2(10\cos^{-1}x)$$

$$= 100\{1-\cos^2(10\cos^{-1}x)\}$$

$$= 100(1-y^2) \qquad \text{[from (1)]}$$

Differentiating again *w.r.t.* x

$$(1-x^2)2y_1y_2 - 2x\cdot y_1^2 = -2\cdot 100y\cdot y_1$$

or, $(1-x^2)y_2 = xy_1 - 100y \qquad (\because\ 2y_1 \neq 0)$ . . . (2)

Differentiating (2) 10 times with the help of Leibnitz's theorem,

$$\left(1-x^2\right)y_{12}+10\cdot y_{11}(-2x)+\frac{10\cdot 9}{1\cdot 2}\cdot y_{10}(-2)$$
$$=xy_{11}+10\cdot y_{10}\cdot(1)-100y_{10}$$

or, $\left(1-x^2\right)y_{12}=21\cdot x\cdot y_{11}$.

Ex. 7. (*i*) If $f(x)=x^n$, prove that

$$f(1)+\frac{f'(1)}{1!}+\frac{f''(1)}{2!}+\frac{f'''(1)}{3!}+\cdots+\frac{f^n(1)}{n!}=2^n$$

(*ii*) If $f(x)=\tan x$ and n is a positive integer, prove with the help of Leibnitz's theorem that

$$f^n(0)-{}^nc_2f^{n-2}(0)+{}^nc_4f^{m-4}(0)-\cdots=\sin\left(n\cdot\frac{\pi}{2}\right)$$ [*C. P. 1992*]

Solution : (*i*) $\because\ f(x)=x^n$

$$f'(x)=nx^{n-1},\quad f''(x)=n(n-1)x^{n-2}$$

$$f^r(x)=n(n-1)(n-2)\cdots(n-r+1)\,x^{n-r},\ \cdots,\ f^n(x)=n!$$

$$\therefore\ f(1)+\frac{f'(1)}{1!}+\frac{f''(1)}{2!}+\frac{f'''(1)}{3!}+\cdots+\frac{f^n(1)}{n!}$$
$$=1+\frac{n}{1!}+\frac{n(n-1)}{2!}+\frac{n(n-1)(n-2)}{3!}+\cdots+\frac{n!}{n!}$$
$$=1+{}^nC_1+{}^nC_2+{}^nC_3+\cdots+{}^nC_n=(1+1)^n=2^n$$

(*ii*) $\because\ f(x)=\tan x=\dfrac{\sin x}{\cos x}$

$$f(x)\cdot\cos x=\sin x \qquad \ldots\quad (1)$$

Applying Leibnitz's theorem to differentiate both the sides n times w.r.t. x, we get

$$f^n(x)\cos x+{}^nC_1f^{n-1}(x)(-\sin x)+{}^nC_2f^{n-2}(x)(-\cos x)$$
$$+{}^nC_3f^{n-3}(x)(\sin x)+{}^nC_4f^{n-4}(x)(\cos x)+\cdots=\sin\left(n\frac{\pi}{2}+x\right)$$

Putting $x=0$ on both the sides,

$$f^n(0)-{}^nC_2f^{n-2}(0)+{}^nC_4f^{n-4}(0)+\cdots\cdots=\sin\left(\frac{n\pi}{2}\right)$$

Ex. 8. If $y=\cos(m\sin^{-1}x)$, show that

(*i*) $\left(1-x^2\right)y_2-xy_1+m^2y=0$

(*ii*) $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}+\left(m^2-n^2\right)y_n=0$

Also, find the value of y_n when $x=0$. [*B. P. 1999*]

Solution : Given, $y=\cos(m\sin^{-1}x)$ ... (1)

Differentiating *w.r.t.* x,

$$\frac{dy}{dx}=y_1=-m\sin\left(m\sin^{-1}x\right)\frac{1}{\sqrt{1-x^2}}$$

or, $\left(1-x^2\right)y_1^2=m^2\sin^2\left(m\sin^{-1}x\right)=m^2\left\{1-\cos^2\left(m\sin^{-1}x\right)\right\}$

or, $\left(1-x^2\right)y_1^2=m^2\left(1-y^2\right)$ [from (1)] ... (2)

Differentiating again *w.r.t.* x

$$\left(1-x^2\right)2y_1\cdot y_2+y_1^2(-2x)=-m^2\cdot 2y\cdot y_1$$

or, $\left(1-x^2\right)y_2-xy_1+m^2y=0,\quad (\because 2y_1\neq 0)$ ... (3)

Differentiating (3) n times by Leibnitz's theorem, we get

$$\left(1-x^2\right)y_{n+2}+{}^nC_1y_{n+1}(-2x)+{}^nC_2y_n(-2)-y_{n+1}\cdot x-{}^nC_1y_n(1)+m^2y_n=0$$

or, $\left(1-x^2\right)y_{n+2}-2n\,xy_{n+1}-n(n-1)y_n-xy_{n+1}-ny_n+m^2y_n=0$

or, $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}+\left(m^2-n^2\right)y_n=0$ ... (4)

Last part : From (1), (2), (3), we have $y=1,\ y_1=0,\ y_2=-m^2$, **when** $x=0$.

putting n = 1, 2, 3 successively in (4), we get

$$y_3=-m^2y_1\ =-m^2\times 0=0$$
$$y_4=\left(2^2-m^2\right)y_2=-m^2\left(2^2-m^2\right)$$
$$y_5=\left(3^2-m^2\right)y_3=0$$
$$y_6=\left(4^2-m^2\right)y_4=-m^2\left(2^2-m^2\right)\left(4^2-m^2\right)$$

Thus, $y_n=0$, when n is odd and,

$y_n=-m^2\left(2^2-m^2\right)\left(4^2-m^2\right)\cdots\left\{(n-2)^2-m^2\right\}$, when n is even.

Ex. 9. If $y = e^{\cos^{-1} x}$, show that an equation connecting y_n, y_{n+1} and y_{n+2} is given by $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+1)y_n = 0$

[*C. P. 1980, B. P. 1996*]

Solution : $y = e^{\cos^{-1} x}$ ... (1)

$\therefore \quad y_1 = e^{\cos^{-1} x} \times \dfrac{-1}{\sqrt{1-x^2}} = -\dfrac{y}{\sqrt{1-x^2}}$, using (1)

or, $(1-x^2)y_1^2 - y^2 = 0$ ... (2)

Differentiating (2) again *w.r.t.* x,

$$(1-x^2)2y_1y_2 + y_1^2(-2x) - 2y \cdot y_1 = 0$$

or, $(1-x^2)y_2 - xy_1 - y = 0$ ... (3)

Differentiating (3) n times with the help of Leibnitz's theorem, we have,

$$(1-x^2)y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - xy_{n+1}$$
$$- {}^nC_1 y_n(1) - y_n = 0$$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+1)y_n = 0$.

Ex. 10. If $x = \sin\theta$, $y = \sin p\theta$, then prove that

(*i*) $(1-x^2)y_2 - xy_1 + p^2y = 0$

(*ii*) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (p^2 - n^2)y_n = 0$

where y_n denotes the *nth* order derivative of y with respect to x.

[*C. P. 2002*]

Solution : Here, $x = \sin\theta$ and $y = \sin p\theta$

Proceeding exactly as in **Ex. 1.** (*ii*), we have

$(1-x^2)y_2 - xy_1 + p^2y = 0$ ... (1)

Differentiating (1) n times with the help of Leibnitz's theorem, we get,

$$(1-x^2)y_{n+2} + n \cdot y_{n+1}(-2x) + \frac{n(n-1)}{1 \cdot 2} \cdot y_n(-2) - xy_{n+1}$$
$$- n \cdot 1 \cdot y_n + p^2 y_n = 0$$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (p^2 - n^2)y_n = 0$.

EXAMPLES - VIII (B)

1. Find y_n in the following cases :

(i) $y = x^2 e^{ax}$. (ii) $y = x^3 \sin x$. (iii) $y = x^3 \log x$.
(iv) $y = x^2 \tan^{-1} x$. (v) $y = e^{ax} \cos bx$. (vi) $y = \log(ax + x^2)$
(vii) $y = x^n(1-x)^n$. (viii) $y = x^n/(1+x)$.

2. If $y = A \sin mx + B \cos mx$, prove that $y_2 + m^2 y = 0$.

3. If $y = A e^{mx} + B e^{-mx}$, prove that $y_2 - m^2 y = 0$.

4. If $y = e^{ax} \sin bx$, show that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.

5. If $y = \log\left(x + \sqrt{a^2 + x^2}\right)$, show that $(a^2 + x^2)y_2 + xy_1 = 0$.

6. If $y = \log\left(x + \sqrt{1 + x^2}\right)^m$, then prove that

$$(1 + x^2)y_2 + xy_1 - m^2 y = 0.$$

7. If $y = \tan^{-1} x$, then prove that

(i) $(1 + x^2)y_1 = 1$, and

(ii) $(1 + x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$.

Find also the value of $(y_n)_0$.

8. If $y = \sin^{-1} x$, then show that

(i) $(1 - x^2)y_2 - xy_1 = 0$,

(ii) $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$. [*C. P. 1997*]

Find also the value of $(y_n)_0$.

9. If $y = (\sin^{-1} x)^2$, then show that

(i) $(1 - x^2)y_2 - xy_1 - 2 = 0$, [*C. P. 1988, 95, 96*]

(ii) $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$.

10. If $\log y = \tan^{-1} x$, then prove that

(i) $(1 + x^2)y_2 + (2x - 1)y_1 = 0$,

(ii) $(1 + x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$.

11. If $y = a\cos(\log x) + b\sin(\log x)$, then prove that.

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$$

[C.P.89,96, B.P.90,97, V.P. 99]

12. If $y = (x^2-1)^n$, then show that

$$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$$

13. If $y = e^{a\sin^{-1}x}$, then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$$

[C.P.1985, 94, 98]

14. If $y = \sin(m\sin^{-1}x)$, then show that

(i) $(1-x^2)y_2 - xy_1 + m^2 y = 0,$ [C.P.1990, 2002]

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$

15. If $y = (ax^2+bx+c)/(1-x)$, show that $(1-x)y_3 = 3y_2$.

16. If $y = e^{-x}\cos x$, prove that $y_4 + 4y = 0.$ [B. P. 1998, 2001]

17. If $y = x^{n-1}\log x$, show that $y_n = \dfrac{(n-1)!}{x}$.

[C.P.1985, 2000, B. P. 1991]

18. Show that $D^r\left(e^{ax}x^n\right) = a^{r-n}x^{n-r}D^n\left(e^{ax}x^r\right)$

19. If u, v, w be functions of x and if suffixes denote differentiations with respect to x, prove that

$$\frac{d}{dx}\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix}.$$

20. By forming in two different ways the nth derivatives of x^{2n}, show that

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2.2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2} + \ldots = \frac{(2n)!}{(n!)^2}$$

[Equate the nth derivative of the product $x^n.x^n$ to that of x^{2n}.]

21. Prove that $\dfrac{d^n}{dx^n}\left(\dfrac{\sin x}{x}\right) = \left\{P\sin\left(x+\dfrac{n\pi}{2}\right) + Q\cos\left(x+\dfrac{n\pi}{2}\right)\right\}/x^{n+1}$

where $P = x^n - n(n-1)x^{n-2} + n(n-1)(n-2)(n-3)x^{n-4} - \ldots$

and $Q = nx^{n-1} - n(n-1)(n-2)x^{n-3} + \ldots$

22. Prove that

$$\frac{d^n}{dx^n}\left(\frac{\cos x}{x}\right) = \left\{P\cos\left(x+\frac{n\pi}{2}\right) - Q\sin\left(x+\frac{n\pi}{2}\right)\right\}\Big/x^{n+1}$$

where P and Q have the same values as in Ex. 21.

23. Prove that

$$\frac{d^n}{dx^n}\left(x^n\sin x\right) = n!\,(P\sin x + Q\cos x),$$

where $P = 1 - \binom{n}{2}\dfrac{x^2}{2!} + \binom{n}{4}\dfrac{x^4}{4!} - \ldots\ldots$

and $Q = \binom{n}{1}x - \binom{n}{3}\dfrac{x^3}{3!} + \binom{n}{5}\dfrac{x^5}{5!} - \ldots\ldots$

24. Show that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = (-1)^n\frac{n!}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right)$$

25. Show that

$$\frac{d^n}{dx^n}\left(e^{-x}x^{n+a}\right) = n!\,e^{-x}x^a\sum_{r=0}^{n}\binom{n+a}{r}\frac{(-x)^{n-r}}{(n-r)!},\quad a > -1.$$

26. If $f(x) = \tan x$, prove that

$$f^n(0) - {}^nc_2 f^{n-2}(0) + {}^nc_4 f^{n-4}(0) - \ldots = \sin\frac{1}{2}n\pi$$

27. Show that the n^{th} differential coefficient of $\dfrac{1}{1+x+x^2+x^3}$ is

$$\frac{1}{2}(-1)^n n!\sin^{n+1}\theta\left\{\sin(n+1)\theta - \cos(n+1)\theta + (\sin\theta+\cos\theta)^{-n-1}\right\}.$$

where $\theta = \cot^{-1}x$.

ANSWERS

1. (i) $e^{ax}.a^{n-2}\left\{a^2x^2 + 2nax + n(n-1)\right\}$

(ii) $x^3 \sin\left(\frac{1}{2}n\pi + x\right) + 3nx^2 \sin\left\{\frac{1}{2}(n-1)\pi + x\right\} + 3n(n-1)x$

$\times \sin\left\{\frac{1}{2}(n-2)\pi + x\right\} + n(n-1)(n-2)\sin\left\{\frac{1}{2}(n-3)\pi + x\right\}.$

(iii) $(-1)^n.6(n-4)!\frac{1}{x^{n-3}}.$

(iv) $(-1)^n(n-3)!\sin^{n-2}\theta\,\{(n-1)(n-2)\sin n\theta\cos^2\theta$
$-2n(n-2)\sin(n-1)\theta\cos\theta + n(n-1)\sin(n-2)\theta\}$ where $\cot\theta = x.$

(v) $e^{ax}\left\{a^n \cos bx + {}^nc_1 a^{n-1} b \cos\left(bx + \frac{1}{2}\pi\right)\right.$

$\left. + {}^nc_2 a^{n-2} b^2 \cos\left(bx + 2.\frac{1}{2}\pi\right) + \dots + b^n \cos\left(bx + n.\frac{1}{2}\pi\right)\right\}$

(vi) $(-1)^{n-1}(n-1)!\left[\frac{1}{x^n} + \frac{1}{(x+a)^n}\right].$

(vii) $n!\left\{(1-x)^n - \left({}^nc_1\right)^2.(1-x)^{n-1}.x + \left({}^nc_2\right)^2.(1-x)^{n-2}.x^2 - \dots\right\}$

(viii) $n!/(1+x)^{n+1}.$

7. 0, or $(-1)^{\frac{1}{2}(n-1)}(n-1)!$ according as n is even or odd.

8. 0, or $\{1.3.5\dots(n-2)\}^2$ according as n is even or odd.